INDIAN STATISTICAL INSTITUTE
Probability Theory II: B. Math (Hons.) I
Semester II, Academic Year 2017-18
Backpaper Exam

Total Marks: 100 **Duration: 3 hours**

- **Please write your roll number on top of your answer paper.**
- **Show all your works and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. A continuous random vector (X, Y) has a joint probability density function given by

$$f_{X,Y}(x,y) = e^{-(x-y)} \quad \text{if } 0 < y < 1 \text{ and } x > y.$$

   (a) (4+4 = 8 marks) Find marginal probability density functions of X and Y.

   (b) (2 marks) Are X and Y independent? Please justify your answer.

   (c) (5+5 = 10 marks) Calculate the conditional probability density functions of X given Y and Y given X.

   (d) (5+5 = 10 marks) Compute $E(X|Y)$ and $Var(Y|X)$.

2. (10 marks) Suppose U and V are two independent random variables with $U, V \sim Unif(0,1)$. Compute the probability density function of $Z = U - V$.

3. Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables having characteristic function $\phi(t) = \exp\{-|t|^{1.8}\}$, $t \in \mathbb{R}$ (assume that this is a valid characteristic function).

   (a) (5 marks) Express the cumulative distribution function of $S_n := X_1 + X_2 + \cdots + X_n$ $(n \geq 1)$ in terms of the cumulative distribution function of X_1.

   (b) (5 marks) Find the weak limit of $n^{-5/8} S_n$ as $n \to \infty$.

4. (10 marks) Let $X_1, X_2, \ldots, X_n$ be i.i.d. standard normal random variables. For $k = 1, \ldots, n-1$, define $Y_k = (\sum_1^k X_i - kX_{k+1})/\sqrt{k(k+1)}$. Then show that $Y_1, \ldots, Y_{n-1}$ are also i.i.d. standard normal random variables.

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5. State whether each of the following statements is true or false. If it is true, give a detailed proof. On the other hand, if it is false, produce a counter-example with full justification. If you both prove and disprove a statement, you will get a zero in that problem.

   (a) (10 marks) If $X_n \xrightarrow{d} X$, then $X_n^2 \xrightarrow{d} X^2$.

   (b) (10 marks) If X and Y are two independent random variables such that at least one of them has a continuous cumulative distribution function, then so does $X+Y$.

   (c) (10 marks) If X_1 and X_2 are i.i.d. random variables following exponential distribution with parameter $\lambda = 2$, then $V := \frac{4X_1+3X_2}{X_1+X_2}$ follows uniform distribution on the interval $(3, 4)$.

6. (10 marks) Find all random variables X satisfying $E(X^2) < \infty$ and the following distributional equation: $X \stackrel{d}{=} (X+Y)/\sqrt{2}$ for any random variable Y independent of X such that $Y \stackrel{d}{=} X$.